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Dipole Solutions in the $SU(2)$ and $SU(3)$ Gauge Theory with Electric and Magnetic Sources

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The magnetic dipole solutions of Sikivie and Weiss are considered with the addition of a magnetic source and the validity of the observation that for large source strengths the energy of such solutions is lower than the energy of corresponding Coulomb solutions is examined. It is found that the presence of electric and magnetic sources leads to dipole solutions and that the introduction of a magnetic source does not alter the relationship between their energy and the energy of corresponding Coulomb solutions.

1. INTRODUCTION

Interest in classical gauge theories has increased in recent years. In this regard, Mandula (1977) first pointed out a peculiar feature of classical Yang-Mills fields in the presence of external sources by showing that the Coulomb field produced by the external sources is unstable if the strength of the sources exceeds a certain critical value. Magg (1978), by considering the external sources as static and spherically symmetric, discussed the stability problem to show that the Coulomb solution of the classical Yang-Mills equations with external sources is unstable for sufficiently strong coupling constant. With an extended charge all spherically symmetric solutions with time-independent fields have also been found (Matheutisch *et al.*, 1982) for the classical Yang-Mills equation in which the energy and charge are reduced compared to the Coulomb solutions. A significant contribution to such studies was made by Sikivie and Weiss (1978), who presented two new classes of solutions to the Yang-Mills field equations in the presence of static, localized, but extended external sources. However, the sources were purely electric ones. In our earlier studies Joshi *et al.* (1985) and Kumar *et al.* (1987) extended the investigations of Sikivie and

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Weiss by including magnetic sources and obtained classical solutions for $SU(2)$ gauge theory with both electric and magnetic sources. The studies were further extended to investigate the classical solution of gauge theories with external electric and magnetic sources for the gauge group $SU(3)$ (Prasad and Joshi, 1989*a*,*b*).

In the present paper we investigate the long-range behavior of $SU(2)$ and $SU(3)$ gauge fields with extended electric and magnetic sources. The magnetic sources chosen here are not of topological origin, but are similar to that introduced by Brandt and Neri (1978) and these sources, in order to avoid the string variables, have necessitated the use of a new non-Abelian field tensor (Joshi *et al.*, 1985; Kumar *et al.*, 1987; Prasad and Joshi, 1989*a*,*b*; Brandt and Neri, 1978; Benjwal and Joshi, 1987). This paper is divided into seven sections. In Section 2, a new non-Abelian field tensor (Benjwal and Joshi, 1987) to describe the electric and magnetic sources is introduced and the field equations and conservation laws are obtained. In Section 3, a description of the extended static electric and magnetic sources in relation to the gauge groups $SU(2)$ and $SU(3)$ is given and in Section 4, cylindrical symmetry is introduced for the fields A_μ and B_μ and the Coulomb solutions for the extended charge distributions are obtained. Section 5 is devoted to the explanation of the long-range behavior of the dipole solutions for both $SU(2)$ and $SU(3)$ gauge groups, and in Section 6, the energy of such solutions is obtained and compared to the energy of the corresponding Coulomb solutions. Concluding remarks are given in Section 7.

2. FIELD EQUATIONS

We consider nontopological electric and magnetic sources and introduce (Benjwal and Joshi, 1987) the following non-Abelian field tensor to describe them:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + ef^{abc} A^b_\mu A^c_\nu - \tfrac{1}{2}\delta_{\mu\nu\rho\sigma}(\partial^\rho B^{\sigma a} - \partial^\sigma B^{\rho a} + gf^{abc} B^{\rho b} B^{\sigma c}) \tag{1}$$

where A^a_μ and B^a_ν are two non-Abelian potentials, e and g are the corresponding gauge coupling parameters, f^{abc} are the structure constants of the gauge group, and $\delta_{\mu\nu\rho\sigma}$ is an antisymmetric tensor. The potentials A^a_μ and B^a_μ obey the gauge transformations

$$A^a_\mu = UA^a_\mu U^{-1} - \frac{1}{e}(\partial_\mu U) U^{-1} \tag{2a}$$

$$B^a_\mu = UB^a_\mu U^{-1} - \frac{1}{g}(\partial_\mu U) U^{-1} \tag{2b}$$

where

$$U = \exp[-i\Lambda^a(x)T^a] \tag{3}$$

in which $\Lambda^a(x)$ with $x = (\mathbf{x}, t)$ are the three independent real functions of space-time, and T represents the group generators of the gauge group obeying

$$[T^a, T^b] = if^{abc}T^c \tag{4}$$

The field tensor (1) transforms as

$$F_{\mu\nu} \to UF_{\mu\nu}U^{-1} \tag{5}$$

The Lagrangian density for the system may be written as

$$L = -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + j^a_\mu A^{\mu a} + k^a_\mu B^{\mu a} \tag{6}$$

in which j^a_μ and k^a_μ are, respectively, the electric and magnetic soucre densities obeying

$$j^{\mu a} \to Uj^{\mu a}U^{-1} \tag{7a}$$

$$k^{\mu a} \to Uk^{\mu a}U^{-1} \tag{7b}$$

The Lagrangian density is invariant under the transformations (2), (5), and (7) and its Euler–Lagrange variations give the following field equations:

$$D_\mu F^{\mu\nu a} = j^{\nu a} \tag{8a}$$

and

$$D'_\mu \tilde{F}^{\mu\nu a} = -k^{\nu a} \tag{8b}$$

where $\tilde{F}^{\mu\nu a}$ is the dual of the field tensor (1),

$$\tilde{F}^{\mu\nu a} = \tfrac{1}{2}\delta_{\mu\nu\rho\sigma}F^{\rho\sigma a} \tag{9}$$

and has the form

$$\tilde{F}^{\mu\nu a} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + gf^{abc}B^b_\mu B^c_\nu + \tfrac{1}{2}\delta_{\mu\nu\rho\sigma}(\partial^\rho A^{\sigma a} - \partial^\sigma A^{\rho a} + ef^{abc}A^{\rho b}A^{\sigma c}) \tag{10}$$

while

$$D_\mu = \partial_\mu + eA^\times_\mu \tag{11a}$$

$$D'_\mu = \partial_\mu + gB^\times_\mu \tag{11b}$$

with $eA^\times_\mu = ef^{abc}A^b_\mu$ and $gB^\times_\mu = gf^{abc}B^b_\mu$, are the covariant derivatives.

It may be noted that although the inclusion of the magnetic source density has required the field tensor in the form (1), both source densities are covariantly conserved, i.e.,

$$D_\nu j^{\nu a} = 0 \tag{12a}$$

and

$$D'_\nu k^{\nu a} = 0 \tag{12b}$$

for which, however, the following relation need be obeyed:

$$F^{ja}_{\mu\nu} \times F^{\mu\nu a} = 0 = F^{ka}_{\mu\nu} \times F^{\mu\nu a} \tag{13}$$

where

$$F^{ja}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + ef^{abc} A^b_\mu A^c_\nu \tag{14a}$$

and

$$F^{ka}_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + gf^{abc} B^b_\mu B^c_\nu \tag{14b}$$

3. THE EXTENDED STATIC SOURCES

The static electric and magnetic sources may be described by

$$j^a_\nu = \delta^0_\nu j^a_0 \tag{15a}$$

and

$$k^a_\nu = \delta^0_\nu k^a_0 \tag{15b}$$

with $j^a_0(x) = q^a_e(x)$ and $k^a_0(x) = q^a_g(x)$ with $x = [\mathbf{x}, t]$. For the static sources (15), the gauge covariance of j^a and k^a [equations (12a) and (12b)] leads to

$$\partial_0 q^a_e(x) = -ef^{abc} A^{0b} q^c_e(x) \tag{16a}$$

and

$$\partial_0 q^a_g(x) = -gf^{abc} B^{0b} q^c_g(x) \tag{16b}$$

These equations imply that the time development of $q_e(\mathbf{x}, t)$ is given by a gauge transformation depending on $A^0(\mathbf{x}, t)$, while that of $q_g(\mathbf{x}, t)$ is given by a gauge transformation depending on $B^0(\mathbf{x}, t)$. The static sources also imply that the Casimir invariants built out of $q^a_e(x)$ and $q^a_g(x)$ are time independent, e.g., for the group $SU(2)$, we have the Casimir invariants

$$C^e(x) = q_e(x) q_e(x) \tag{17a}$$

$$C^g(x) = q_g(x) q_g(x)' \tag{17b}$$

which, on using equations (16a) and (16b), can be seen to be time independent, i.e.,

$$\partial_0 C_e = \partial_0 C_g = 0 \tag{18}$$

We now consider a system of electric and magnetic sources which have no δ-function singularities, but instead have spherically symmetric charge distributions

$$q_e(r) = q_e(x) = C \exp(-cr) \tag{19a}$$

$$q_g(r) = q_g(x) = D \exp(-dr) \tag{19b}$$

for $r = |x| \to \infty$, where C and D are constants and c and d are positive quantities. Since equations (19a) and (19b) imply that the electric and magnetic charge distributions have infinite extensions, the total electric and magnetic charges are given by

$$Q_e = \int_0^\infty 4\pi r^2 q_e(r)\, dr \tag{20a}$$

$$Q_g = \int_0^\infty 4\pi r^2 q_g(r)\, dr \tag{20b}$$

The total charge (20) may, however, be looked upon as the sum of the charge between the origin and a certain radius r_0 and the charge between r_0 and ∞. If the fractions of total electric and magnetic charges outside the radius r_0 are $P_e(r)$ and $P_g(r)$, respectively, we may write

$$P_e(r) = \frac{1}{Q_e} \int_r^\infty 4\pi r^2 q_e(r)\, dr \tag{21a}$$

$$P_g(r) = \frac{1}{Q_g} \int_r^\infty 4\pi r^2 q_g(r)\, dr \tag{21b}$$

The implication of equations (19a) and (19b) that for $r \to \infty$, $q(r) \to 0$ also suggests from equations (21a) and (21b) that

$$q_e(r) = -\frac{Q_e}{4\pi r^2} \frac{dp_e(r)}{dr} \tag{22a}$$

$$q_g(r) = -\frac{Q_g}{4\pi r^2} \frac{dp_g(r)}{dr} \tag{22b}$$

We now review equation (18) and examine its implications. The equation implies that the static sources which construct time-independent Casimir invariants are themselves time independent. For the time-independent sources

$$\partial_0 q_e^a = 0 \tag{23a}$$

$$\partial_0 q_g^a = 0 \tag{23b}$$

equations (16a) and (16b) read

$$f^{abc}A_0^b q_e^c = 0 \tag{24a}$$

$$f^{abc}B_0^b q_g^c = 0 \tag{24b}$$

From equation (24a), it is observed that if q_e is considered to be lined up in the commuting directions of the gauge group space, i.e., for all x and x',

$$[q_e(x), q_e(x')] = 0 \tag{25a}$$

then A_0 is also aligned along the commuting directions of the gauge group space. Similarly equation (24b) and

$$[q_g(x), q_g(x')] = 0 \tag{25b}$$

align B_0 the way q_g is aligned. Thus, conversely, equations (23a) and (23b) suggest that by specifying A_0 and B_0, we specify the sources q_e and q_g, respectively. Equation (18) also implies that the source distributions only rotate in the gauge group space. Therefore, recalling Zwanziger's (1968) generalized charge vector whose two vector components represent the electric and magnetic charges, it may be assumed for the gauge group $SU(2)$ that the electric charge rotates about the δ^{a1} axis and the magnetic charge rotates about the δ^{a3} axis in the isospin space, such that the general electric and magnetic charge distributions are given by

$$q_e^a(r) = q_e(r)(\delta^{a3}\cos\theta + \delta^{a2}\sin\theta) \tag{26a}$$

$$q_g^a(r) = q_g(r)(\delta^{a1}\cos\theta' + \delta^{a2}\sin\theta') \tag{26b}$$

where $\theta = 2\pi n p_e(r)$ and $\theta' = 2\pi n p_g(r)$. However, the electric and magnetic charges may be gauge rotated such that they align themselves along the δ^{a3} and δ^{a1} axes, respectively (Kumar *et al.*, 1987), i.e.,

$$q_e^a(r) = q_e(r)\delta^{a3} \tag{27a}$$

$$q_g^a(r) = q_g(r)\delta^{a1} \tag{27b}$$

Viewing these equations along with equations (17a) and (17b), we observe that the Casimir invariants (17a) and (17b) for the gauge group $SU(2)$ remain unchanged if δ^{a3} and δ^{a1} are locally changed to $-\delta^{a3}$ and $-\delta^{a1}$, respectively. Therefore, in $SU(2)$ a source distribution aligned along δ^{ai} may be locally changed to $-\delta^{ai}$ $(i = 1, 2, 3)$.

In the case of gauge group $SU(3)$, since it is of rank 2, it has, contrary to $SU(2)$, the following Casimir invariants:

$$C_1^e(x) = [q_e^3(x)]^2 + [q_e^8(x)]^2 \tag{28a}$$

$$C_1^g(x) = [q_g^3(x)]^2 + [q_g^8(x)]^2 \tag{28b}$$

and

$$C_2^e(x) = -\frac{1}{\sqrt{3}}[q_e^8(x)]^3 + \sqrt{3}[q_e^3(x)]^2 q_e^8(x) \tag{29a}$$

$$C_2^g(x) = -\frac{1}{\sqrt{3}}[q_g^8(x)]^3 + \sqrt{3}[q_g^3(x)]^2 q_g^8(x) \tag{29b}$$

$$q(x) = -\tfrac{1}{2}\lambda^a q^a(x) \tag{30}$$

where $q(x)$ stands for both $q_e(x)$ and $q_g(x)$ and λ^a, the generators for the gauge group $SU(3)$, are denoted by the Gell-Mann matrices, $a = 1, 2, \ldots, 8$; only λ^3 and λ^8 are diagonal. It is easy to observe that the other set of Casimir invariants [equations (29a) and (29b)] changes when the changes $\delta^{a8} \to -\delta^{a8}$ and $\delta^{a3} \to -\delta^{a3}$ are made. Therefore, in this case the source distributions aligned along $\delta^{aa'}$ ($a' = 3, 8$) cannot locally be changed into $-\delta^{aa'}$. This is a significant departure from the corresponding case in $SU(2)$. However, when $C_2^e = C_2^g = 0$, equations (28a) and (28b) reduce to their $SU(2)$ counterparts [equations (27a) and (27b)]

$$C_1^e = [q_e^3]^2 \tag{31a}$$

$$C_1^g = [q_g^3]^2 \tag{31b}$$

implying that the electric and magnetic charges may be gauge rotated only to the δ^{a3}, and it then lies in the $SU(2)$ subalgebra of $SU(3)$. When C_2^e and C_2^g are not vanishing, the sources can be written as

$$q_e^a(x) = \delta^{a3} q_e^3(x) + \delta^{a8} q_e^8(x) \tag{32a}$$

and

$$q_g^a(x) = \delta^{a3} q_g^3(x) + \delta^{a8} q_g^8(x) \tag{32b}$$

4. CYLINDRICALLY SYMMETRIC FIELDS AND COULOMB SOLUTIONS

We now simplify the field equations (8) by imposing cylindrical symmetry for the fields A_μ and B_μ and assume that both the sources and the fields are time independent, i.e., the sources obey equations (23a) and (23b) fields

$$\partial_0 A_\mu^a = 0 \tag{33a}$$

$$\partial_0 B_\mu^a = 0 \tag{33b}$$

Then, we may write from the field tensors (1) and (10) that

$$F^{0ia} = -\partial^i A^{0a} + e(A^0 \times A^i)^a - \varepsilon_{ijk}[(\partial_j B_k - \partial_k B_j) + g(B_j \times B_k)]^a \tag{34a}$$

and

$$\tilde{F}^{0ia} = -\partial^i B^{0a} + g(B^0 \times B^i)^a - \varepsilon_{ijk}[(\partial_j A_k - \partial_k A_j) + e(A_j \times A_k)]^a \quad (34b)$$

Consequently, the field equations (8a) and (8b) acquire the following forms under the conditions (33a) and (33b):

$$D_i F^{0ia} = -\partial_i \partial^i A^{0a} + 2e(\partial_i A^0 \times A^i)^a + e(A^0 \times \partial_i A^i)^a + e^2[A^i \times (A^0 \times A^i)]^a + \varepsilon_{ijk} g[(\partial_i B^j \times B^k)^a + (B^j \times \partial_i B^k)^a] = q_e^a(x) \quad (35a)$$

$$D_i \tilde{F}^{0ia} = -\partial_i \partial^i B^{0a} + 2g(\partial_i B^0 \times B^i)^a + g(B^0 \times \partial_i B^i)^a + g^2[B^i \times (B^0 \times B^i)]^a + \varepsilon_{ijk} e[(\partial_i A^j \times A^k) + (A^j \times \partial_i A^k)]^a = -q_g^a(x) \quad (35b)$$

$$D_0 F^{0ja} = e\{A_0 \times [-\partial^j A^0 + e(A^0 \times A^j)]\}^a \quad (36a)$$

$$D_0' \tilde{F}^{0ja} = g\{B_0 \times [-\partial^j B^0 + g(B^0 \times B^j)]\}^a \quad (36b)$$

Using equations (24a) and (24b), the first terms in equations (36a) and (36b) may be seen to vanish, leaving

$$D_0 F^{0ja} = e^2[A_0 \times (A^0 \times A^j)]^a \quad (37a)$$

and

$$D_0' \tilde{F}^{0ja} = g^2[B_0 \times (B^0 \times B^j)]^a \quad (37b)$$

respectively. The field equations (8a) and (8b) with static sources also lead to the relations

$$D_0 F^{0ja} = D_i F^{ija} = \text{RHS of (37a)} \quad (38a)$$

$$D_0' \tilde{F}^{0ja} = D_i' \tilde{F}^{ija} = \text{RHS of (37b)} \quad (38b)$$

We now impose cylindrical symmetry on the fields and first consider the case of the gauge group $SU(2)$, for which we assume that the field A_μ has cylindrical symmetry around the $\hat{3}$ axis and B_μ has it around the $\hat{1}$ axis. In view of equations (24a) and (24b), similar directions are set for the electric and magnetic sources as well. Following these assumptions, we may make the ansatz (Sikivie and Weiss, 1978)

$$A_0 = \phi_e(\rho, x_3), \qquad A_i = \varepsilon_{i3k} \frac{x_k}{\rho} A(\rho, x_3) \quad (39a)$$

and

$$B_0 = -\phi_g(\rho', x_1'), \qquad B_i = -\varepsilon_{i1k} \frac{x_k'}{\rho'} B(\rho', x_1') \quad (39b)$$

where

$$\rho = (x_1^2 + x_2^2)^{1/2}, \qquad \rho' = (x_2'^2 + x_3'^2)^{1/2} \quad (39c)$$

Consequently,

$$\partial_i A_i = 0, \qquad \partial_i B_i = 0 \tag{40a}$$

and

$$A_i \partial_i A_0 = 0, \qquad B_i \partial_i B_0 = 0 \tag{40b}$$

It may be noted that the minus sign in equation (39b) is due to the minus sign in the field equation (8b). This sign in fact implies that the cylindrical symmetry for the magnetic source is around the negative $\hat{1}$ axis.

Using equations (39) and (40), we can write equations (35) as

$$-\nabla^2 \phi_e + e^2 A \times (\phi_e \times A) = q_e^3(\mathbf{x}) \tag{41a}$$

$$-\nabla^2 \phi_g + g^2 B \times (\phi_g \times B) = q_g^1(\mathbf{x}) \tag{41b}$$

We may observe from equation (39a) that ϕ_e is directed along the $\hat{3}$ axis in the isospin space, while A_3 is vanishing, and from (39b), that ϕ_g is directed along the $\hat{1}$ axis with B_1 vanishing. Thus, from equations (40a) and (41a), we have for A_1 and A_2

$$-\nabla^2 \phi_e^3 + e^2 \phi_e^3[(A_1)^2 + (A_2)^2] = q_e^3(\mathbf{x}) \tag{42a}$$

$$\left[\nabla^2 - \frac{1}{\rho^2} + (e\phi_e^3)^2\right] A_{1,2} = 0 \tag{42b}$$

$$\phi_3^3 A_2 A_3 = \phi_e^3 A_1 A_3 = 0 \tag{43a}$$

$$\nabla^2 A_3 - \frac{1}{\rho^2} A_3 = 0 \tag{43b}$$

Similarly, from (40b) and (41b), we have for B_2 and B_3

$$-\nabla^2 \phi_g^1 + g^2 \phi_g^1[(B_2)^2 + (B_3)^2] = q_g^1(\mathbf{x}') \tag{44a}$$

$$\left[\nabla^2 - \frac{1}{\rho'^2} + (g\phi_g^1)^2\right] B_{2,3} = 0 \tag{44b}$$

$$\phi_g^1 B_2 B_1 = \phi_g^1 B_3 B_1 = 0 \tag{45a}$$

$$\nabla^2 B_1 - \frac{1}{\rho'^2} B_1 = 0 \tag{45b}$$

In equation (42a), A_1 and A_2 are interchangeable and so are B_2 and B_3 in equation (44a). Therefore, for simplicity we may assume that $A_2 = 0 = B_2$. Moreover, equations (43b) and (45b) demand that in order to observe the boundary conditions $A_3 \to 0$, $B_1 \to 0$ at infinity both A_3 and B_1 should vanish. We then have only the following equations for (42)-(45):

$$-\nabla^2 \phi_e^3 + e^2 \phi_e^3 (A_1)^2 = q_e^3(\mathbf{x}) \tag{46a}$$

$$\nabla^2 A_1 - \frac{1}{\rho^2} A_1 + e^2 (\phi_e^3)^2 A_1 = 0 \tag{46b}$$

and

$$-\nabla^2 \phi_g^1 + g^2 \phi_g^1 (B_3)^2 = q_g^1(\mathbf{x}') \tag{47a}$$

$$\nabla^2 B_3 - \frac{1}{\rho'^2} B_3 + g^2 (\phi_g^1)^2 B_3 = 0 \tag{47b}$$

It is therefore observed that for the static time-independent electric and magnetic sources, the field equations (8a) and (8b) have assumed the forms (46) and (47) under the imposition of cylindrical symmetry. It may be noted that in these equations when $A_1 = 0 = B_3$ they reduce to the simple forms

$$-\nabla^2 \phi_e^3 = q_e^3(\mathbf{x}) \tag{48a}$$

$$-\nabla^2 \phi_g^1 = q_g^1(\mathbf{x}') \tag{48b}$$

and the corresponding solutions are the Coulomb ones.

For the gauge group $SU(3)$, the field equations are again (35)-(38), except that the structure constants in the cross products are those of $SU(3)$. The ansatz corresponding to equations (39a)-(39c) in this case is

$$A_0 = \phi_e(\rho, x_l), \qquad A_i = \varepsilon_{ilj} \frac{x_j}{\rho} A(\rho, x_l) \tag{49a}$$

$$B_0 = -\phi_g(\rho', x_l), \qquad B_i = -\varepsilon_{ilj} \frac{x_j'}{\rho'} B(\rho', x_1') \tag{49b}$$

where

$$\rho = (x_i^2 + x_j^2)^{1/2}, \qquad \rho' = (x_i'^2 + x_j'^2)^{1/2} \tag{49c}$$

l can take values 3 and 8, while i, j have values 1, 2, 4, 5, 6, and 7. ϕ_e, ϕ_g, A, and B are now

$$\phi_e = -\tfrac{1}{2} i (\phi_e^3 \lambda_3 + \phi_e^8 \lambda^8) \tag{50a}$$

$$\phi_g = -\tfrac{1}{2} i (\phi_g^3 \lambda_3 + \phi_g^8 \lambda^8) \tag{50b}$$

$$A = -\tfrac{1}{2} i \sum_{k=1,2,4,5,6,7} A_k \lambda_k \tag{51a}$$

$$B = -\tfrac{1}{2} i \sum_{k=1,2,4,5,6,7} B_k \lambda_k \tag{51b}$$

Now, from the discussion below equation (30), we may recall that no group transformation can change $+\delta^{a8}$ to $-\delta^{a8}$; however, δ^{a8} can be changed to (Sikivie and Weiss, 1978)

$$\delta^{a8} \to -\tfrac{1}{2}\delta^{a8} + \frac{\sqrt{3}}{2} \sum_{i=1}^{3} \alpha_i \delta^{ai} \tag{52a}$$

with

$$\sum_i |\alpha_i|^2 = 1 \tag{52b}$$

which in turn lead to the following transformations for δ^{a8}:

$$\delta^{a8} \to \delta^{a8} \tag{53a}$$

$$\delta^{a8} \to -\tfrac{1}{2}\delta^{a8} + \frac{\sqrt{3}}{2}\delta^{a3} \tag{53b}$$

$$\delta^{a8} \to -\tfrac{1}{2}\delta^{a8} - \frac{\sqrt{3}}{2}\delta^{a3} \tag{53c}$$

Corresponding to these transformations, the three components of the source $(-\psi^\dagger(x)\psi(x)\delta^{a8})$ produced by a ψ in the triplet representation are given by

$$\psi = \psi \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad q_{\mathrm{I}}^a = -\delta^{a8} \tag{54a}$$

$$\psi = \psi \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad q_{\mathrm{II}}^a = \frac{\delta^{a8}}{2} - \frac{\sqrt{3}}{2}\delta^{a3} \tag{54b}$$

$$\psi = \psi \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad q_{\mathrm{III}}^a = \frac{\delta^{a8}}{2} + \frac{\sqrt{3}}{2}\delta^{a3} \tag{54c}$$

Similarly, the three components of the source $(-\psi^\dagger(x)\psi(x)\delta^{a3})$ in the triplet representation are

$$-\delta^{a3}, \frac{\delta^{a3}}{2} \pm \frac{\sqrt{3}}{2}\delta^{a8} \tag{55}$$

Using equation (49)-(51) in the field equations (35)-(38) for the source components (54) and (55), we obtain the following set of equations for the three cases.

Case I. All the A_k and B_k vanish except A_1, A_2 and B_1, B_2:

$$-\nabla^2\phi_e^3+e^2[(A_1)^2+(A_2)^2]\phi_e^3=q_e^3(x) \tag{56a}$$

$$-\nabla^2\phi_e^8=q_3^8(x) \tag{56b}$$

$$\left[\nabla^2-\frac{1}{\rho^2}+e^2(\phi_e^3)^2\right]A_{1,2}=0 \tag{56c}$$

$$-\nabla^2\phi_g^3+g^2[(B_1)^2+(B_2)^2]\phi_g^3=q_g^3(x) \tag{57a}$$

$$-\nabla^2\phi_g^8=q_g^8(x) \tag{57b}$$

$$\left[\nabla^2-\frac{1}{\rho'^2}+g^2(\phi_g^3)^2\right]B_{1,2}=0 \tag{57c}$$

Case II. All A_k and B_k vanish except for A_4, A_5 and B_4, B_5:

$$-\nabla^2\left(\frac{\phi_e^3+\sqrt{3}\,\phi_e^8}{2}\right)+e^2[(A_4)^2+(A_5)^2]\frac{\phi_e^3+\sqrt{3}\,\phi_e^8}{2}=\frac{q_e^3+\sqrt{3}\,q_e^8}{2} \tag{58a}$$

$$-\nabla^2\left(-\frac{\sqrt{3}\,\phi_e^3+\phi_e^8}{2}\right)=-\frac{\sqrt{3}\,q_e^3+q_e^8}{2} \tag{58b}$$

$$\left(\nabla^2-\frac{1}{\rho^2}+e^2\frac{\phi_e^3+\sqrt{3}\,\phi_e^8}{2}\right)A_{4,5}=0 \tag{58c}$$

and

$$-\nabla^2\left(\frac{\phi_g^3+\sqrt{3}\,\phi_g^8}{2}\right)+g^2[(B_4)^2+(B_5)^2]\frac{\phi_g^3+\sqrt{3}\,\phi_g^8}{2}=\frac{q_g^3+\sqrt{3}\,q_g^8}{2} \tag{59a}$$

$$-\nabla^2\left(-\frac{\sqrt{3}\,\phi_g^3+\phi_g^8}{2}\right)=-\frac{\sqrt{3}\,q_g^3+q_g^8}{2} \tag{59b}$$

$$\left(\nabla^2-\frac{1}{\rho'^2}+g^2\frac{\phi_g^3+\sqrt{3}\,\phi_g^8}{2}\right)B_{4,5}=0 \tag{59c}$$

Case III. All A_k and B_k vanish except A_6, A_7 and B_6, B_7:

$$-\nabla^2\left(\frac{\phi_e^3-\sqrt{3}\,\phi_e^8}{2}\right)+e^2[(A_6)^2+(A_7)^2]\frac{\phi_e^3-\sqrt{3}\,\phi_e^8}{2}=\frac{q_e^3-\sqrt{3}\,q_e^8}{2} \tag{60a}$$

$$-\nabla^2\left(\frac{\sqrt{3}\,\phi_e^3+\phi_e^8}{2}\right)=\frac{\sqrt{3}\,q_e^3+q_e^8}{2} \tag{60b}$$

$$\left(\nabla^2-\frac{1}{\rho^2}+e^2\frac{\phi_e^3-\sqrt{3}\,\phi_e^8}{2}\right)A_{6,7}=0 \tag{60c}$$

and

$$-\nabla^2\left(\frac{\phi_g^3-\sqrt{3}\,\phi_g^8}{2}\right)+g^2[(B_6)^2+(B_7)^2]\frac{\phi_g^3-\sqrt{3}\,\phi_g^8}{2}=\frac{q_g^3-\sqrt{3}\,q_g^8}{2} \tag{61a}$$

$$-\nabla^2\left(\frac{\sqrt{3}\,\phi_g^3+\phi_g^8}{2}\right)=\frac{\sqrt{3}\,q_g^3+q_g^8}{2} \tag{61b}$$

$$\left(\nabla^2-\frac{1}{\rho^2}+g^2\frac{\phi_g^3-\sqrt{3}\,\phi_g^8}{2}\right)B_{6,7}= \tag{61c}$$

If we set $A=0$ and $B=0$ in each of the above three cases, they all become similar and the corresponding solutions are the Coulomb ones.

5. THE DIPOLE SOLUTIONS AND THE LONG-RANGE BEHAVIOR

The Coulomb solutions, as mentioned above, are obtained by avoiding the nonlinear terms in the field equations, which has been achieved by setting $A=0$ and $B=0$. However, solutions of the field equations with nonlinear terms may also be obtained. Such solutions may be separated into two classes, the short-range and the dipole solutions. We obtain and discuss them in the following for both $SU(2)$ and $SU(3)$ gauge groups using the Sikivie–Weiss (1978) approach.

For the gauge group $SU(2)$, the field equations are (46) and (47) and we assume that for r, $r'\to 0$ both $A_1(\rho, x_3)$ and $B_3(\rho', x_1)$ tend to zero and far away from the origin $A_1\to\alpha$ and $B_3\to\beta$ such that the field equations (46) and (47) approach

$$\nabla^2\alpha-\frac{1}{\rho^2}\alpha=0 \tag{62a}$$

and

$$\nabla^2\beta-\frac{1}{\rho'^2}\beta=0 \tag{62b}$$

The first condition ensures the integrability of the energy density at the origin, while the second condition implies the exponentially fast vanishing of both ϕ_e and ϕ_g. Therefore, the conditions preceding equations (62) on A_1 and B_3 help determine the solutions of the field equations (46) and (47) for specific electric and magnetic sources. The specific character of the sources may be understood from the following. Let us review equations (46a) and (46b). For given A_1 which obeys the condition (62a) and that preceding it, equation (45b) is solved for ϕ_e^3. Using this ϕ_e^3 and the already given A_1, equation (45a) is solved to obtain $q_e^3(x)$. Thus, for this specific $q_e^3(x)$, A_1 and ϕ_e^3 provide the solutions. Similarly, from equations (47a) and (47b) B_3 and ϕ_g^1 give the solutions for magnetic source distribution $q_g^1(x)$ obeying (47a). It therefore appears that equations (46) and (47) do not provide solutions for an arbitrary charge distribution, but do for the specific electric and magnetic charge distributions obeying (46a) and (47a), which themselves depend on (46b) and (47b), respectively.

In order to realize the conditions imposed on A_1 and B_3 we choose particular solutions to equations (62). These solutions must have the properties of ensuring the finiteness of the energy at infinity and the localization of charge at the origin. The simplest solutions which obey these properties are

$$\alpha = \frac{\sin\theta}{r^2} \tag{63a}$$

and

$$\beta = \frac{\sin\theta'}{r'^2} \tag{63b}$$

Therefore, the fields $A_1(\rho, x_3)$ and $B_3(\rho', x_1')$ which obey the requisite conditions have the form

$$A_1(\rho, x_3) = Ca\frac{\sin\theta}{r^2} f\left(\frac{r}{a}, \theta\right) \tag{64a}$$

where C is the norm of $A_1(\rho, x_3)$, a is a parameter depicting the spatial extension of the charge distribution q_e and $f(r/a, \theta)$ is the shape function corresponding to q_e: $f(r/a, \theta) \to 1$ when $r \to \infty$. Similarly, the field $B_3(\rho', x_1')$ has the form

$$B_3(\rho', x_1') = C'a'\frac{\sin\theta'}{r'^2} f\left(\frac{r'}{a'}, \theta'\right) \tag{64b}$$

where c', a', and f' are the norm of $B_3(\rho', x_1')$, the spatial extension of q_g, and shape function for q_g, respectively. Now using (64a) in equation (46b), we may obtain

$$e\phi_e^3 = \frac{1}{a} F(x, \theta) \tag{65a}$$

where

$$F(x, \theta) = \left[-\frac{1}{f}\left(\nabla^2 f - \frac{4}{r}\frac{\partial f}{\partial r} + \frac{2}{r^2 \tan\theta}\frac{\partial f}{\partial \theta}\right)\right]^{1/2} \tag{65b}$$

with $x = r/a$. Similarly, using (64b) in equation (47b), we obtain

$$g\phi_g^1 = \frac{1}{a'} F'(x', \theta') \tag{66a}$$

where

$$F'(x', \theta') = \left[-\frac{1}{f'} \left(\nabla^2 f' - \frac{4}{r'} \frac{\partial f'}{\partial r'} + \frac{2}{r'^2 \tan \theta'} \frac{\partial f}{\partial \theta'} \right) \right]^{1/2} \tag{66b}$$

with $x' = r'/a'$. The total charge is

$$Q = Q_e + Q_g \tag{67}$$

where

$$\begin{aligned} Q_e &= \int q_e^3(x)\, d^3x = \int d^3x [-\nabla^2 \phi_e^3 + e^2 (A_1)^2 \phi_e^3] \\ &= \int_0^\infty \int_0^\pi \int_0^{2\pi} (x^2 \sin \theta \, dx \, d\theta \, d\phi) e^2 (A_1)^2 \phi_e^3 \\ &= eC^2 I_1 \end{aligned} \tag{68}$$

where

$$I_1 = 2\pi \int_0^\pi \sin^3 \theta \, d\theta \int_0^\infty \frac{dx}{x^2} f^2(x, \theta) F(x, \theta) \tag{69}$$

Similarly, Q_g is

$$\begin{aligned} Q_g &= \int d^3x [-\nabla^2 \phi_g^1 + g^2 (B_3)^2 \phi_g^1] \\ &= gc'^2 I_1' \end{aligned} \tag{70}$$

where I_1' may be obtained from I_1 by substituting $\theta \to \theta'$, $x \to x'$, $f \to f'$, and $F \to F'$.

Now in order to obtain the field strength we observe equations (46a) and (47a). From equation (46a) we observe that the electric charge distribution $q_e^3(x)$ takes contributions from the terms $-\nabla^2 \phi_e^3$ and $e^2(A_1)^2 \phi_e^3$ and similarly the magnetic charge distribution $q_g^1(x)$ is a sum of $-\nabla^2 \phi_g^1$ and $g^2(B_3)^2 \phi_g^1$. The contributions $e(A_1)^2 \phi_e^3$ and $g(B_3)^2 \phi_g^1$ appear as a result of the interaction between A_1 and ϕ_e^3 and between B_3 and ϕ_g^1, respectively.

From these contributions we therefore obtain the field strengths as

$$E^{1e}=0$$

$$E^{2e}=e\phi_e^3A_1=eF(x,\theta)f(x,\theta)\frac{\hat{3}\times\hat{1}}{r^2}$$

$$E^{3e}=-\nabla\phi_e^3$$

$$E^{2g}=(-\nabla\times\mathbf{B}_3)$$

$$=\frac{-3(\mathbf{p}_1\cdot\mathbf{x}')\mathbf{x}'-\mathbf{p}_1r'^3}{r'^5}-\frac{\mathbf{p}_1(\mathbf{x}'\cdot\nabla f'-\mathbf{x}'(\mathbf{p}_1\cdot\nabla f')}{r'^3} \tag{71}$$

where $p_1=c'a'\hat{1}$ is the electric dipole moment and E^e is the electric field due to electric source and E^g is the electric field due to the magnetic source. The magnetic field strengths may be obtained as

$$H^{1e}=(\nabla\times\mathbf{A}_1)$$

$$=\frac{3(\mathbf{m}_1\cdot\mathbf{x})\mathbf{x}-\mathbf{m}_1r^3}{r^5}f\left(\frac{r}{a},\theta\right)+\frac{\mathbf{m}_1(\mathbf{x}\cdot\nabla f)-\mathbf{x}(\mathbf{x}\cdot\nabla f)}{r^3} \tag{72}$$

$$H^{2e}=0,\qquad H^{3e}=0$$

$$H^{1g}=-\nabla\phi_g^1$$

$$H^{2g}=g\phi_g^1B_3=gF'(x',\theta')f'(x',\theta')\frac{\hat{3}\times\hat{1}}{r'^2} \tag{73}$$

$$H^{3g}=0$$

where $m_1=Ca\hat{3}$ is the magnetic dipole moment. Combining these equations, we can write the total electric and magnetic fields as

$$\mathbf{E}=-\nabla\phi_e^3+e\phi_e^3A_1-(\nabla\times\mathbf{B}_3) \tag{74a}$$

$$\mathbf{H}=-\nabla\phi_g^1+g\phi_g^1B_3+(\nabla\times\mathbf{A}_1) \tag{74b}$$

It may be observed that only the field strengths E^{2g} and H^{1e} are long ranged. The field strengths other than E^{2e}, E^{3e}, H^{1g}, and H^{2g} are vanishing. For the electric fields E^{3e} and E^{2e} it may be noted that they are produced by charge distributions $-\nabla^2\phi_e^3$ and $e(A_1)^2\phi_e^3$, respectively, and the total charge, equation (67) or (42a), shows that if q_e^3 does not change sign, the charge distribution due to nonlinear terms will always provide screening. The total screening effect thus diminishes the range of electric fields. Similarly, the total screening of magnetic source $q_g^1(x)$ by the charge distributions due to the nonlinear terms $(gB_3)^2\phi_g^1$ makes the magnetic fields short ranged. The electric field of E^{2g} has a resemblance to the long-range behavior of an electric dipole field, while the magnetic field H^{1e} resembles the long-range behavior of a magnetic dipole.

In the case of gauge group $SU(3)$, the field equations are (56)–(61) and their solution may be obtained by following an approach similar to that for $SU(2)$. However, in $SU(3)$, there are three cases, depending upon the vanishing components of A_k and B_k. We obtain in the following the solutions to the field equations for these three cases. We assume that the components of A_k and B_k appearing in each of the three cases obey the conditions (62) and those preceding them. We retain the notation of equations (64) for these nonvanishing components in each case.

Case I. In this case, out of all the components of A_k and B_k, only A_1, A_2 and B_1, B_2 are nonvanishing and the corresponding field equations are (56) and (57). As in $SU(2)$, A_2 and B_2 may be taken as vanishing and A_1 and B_1 obeying (64a) and (64b), respectively. The total charge therefore is

$$Q = Q_e + Q_g \tag{75}$$

where

$$\begin{aligned} Q_e &= \int d^3x[-\nabla^2(\phi_e^3 + \phi_e^8) + e^2\phi_e^3(A_1)^2] \\ &= \int d^3x\, e^2\phi_e^3(A_1)^2 \\ &= eC^2 I_1 \end{aligned} \tag{76a}$$

and

$$\begin{aligned} Q_g &= \int d^3x[-\nabla^2(\phi_g^8 + \phi_g^3) + g^2\phi_g^3(B_1)^2] \\ &= \int d^3x\, g^2\phi_g^3(B_1)^2 = ec'^2 I_1' \end{aligned} \tag{76b}$$

where I_1 is given by equation (69) and I_1' is the primed integral of I_1. Further, from the field equations (56), (57) and the discussion below equation (70), it is observed that the sources q_e^8 and q_g^8 produce electric and magnetic Coulomb fields, while q_e^3 and q_g^3 produce the magnetic and electric dipole fields, respectively. The electric field strengths are

$$E^{3e} = -\nabla\phi_e^3, \qquad E^{8e} = -\nabla\phi_e^8$$

$$E^{2e} = e\phi_e^3 A_1 = eF(x, \theta) f(x, \theta)\frac{\hat{1}\times\hat{3}}{r^2}$$

$$\begin{aligned} E^{1g} &= -(\nabla\times\mathbf{B}_1) \\ &= -\frac{3(\mathbf{p}_2\cdot\mathbf{x}')\mathbf{x}' - \mathbf{p}_2 r'^3}{r'^5} f\left(\frac{r'}{a'}, \theta'\right) - \frac{\mathbf{p}_2(\mathbf{x}'\cdot\nabla f') - \mathbf{x}'(\mathbf{p}_2 - \nabla f')}{r'^3} \end{aligned} \tag{77}$$

The magnetic field strengths are

$$H^3g = -\nabla\phi_g^3, \qquad H^{8q} = -\nabla\phi_g^8$$

$$H^{2g} = g\phi_g^3 B_1 = gF(x', \theta')f(x', \theta')\frac{\hat{1}\times\hat{3}}{r'^2}$$

$$H^{1e} = (\nabla\times\mathbf{A}_1)$$

$$= \frac{3(\mathbf{m}_2\cdot\mathbf{x}) - \mathbf{m}_2 r^3}{r^5} f\left(\frac{r}{a}, \theta\right) + \frac{\mathbf{m}_2(\mathbf{x}\cdot\nabla f) - \mathbf{x}(\mathbf{m}_2\cdot\nabla f)}{r^3} \tag{78}$$

In these equations $p_2 = C'a'\hat{3}$ is the electric dipole moment and $m_2 = Ca\hat{3}$ is the magnetic dipole moment. It may be noted that E^{1g} has the long-range behavior of the magnetic dipole field and H^{1e} has that of the electric dipole field. The other field strengths in equations (77) and (78) are short ranged. The reason may be sought in equations (76a) and (76b), which imply that the charge distribution generated by the gauge field A_1 and ϕ_e^3 as well as that by B_1 and ϕ_g^3 exactly cancel the total charges Q_e and Q_g, respectively. The effect of these charge distributions may also be seen with respect to field equations (56a) and (57a), where it is easily observed that if $q_e^3(x)$ and $q_g^3(x)$ do not change sign, the charge distributions screen these sources and, contrary to short-range phenomena, the screening is partial. Equations (56b) and (57b) tell us that q_e^8 and q_g^8 produce Coulomb fields. Therefore we have three kinds of solutions: those giving Coulomb fields; totally screened ones, which create short-range fields; and partially screened ones, which show a long-range behavior of dipole fields. Since for this case A_1, A_2 and B_1, B_2 are nonvanishing, they may be seen to form two members of I spin (Huang, 1982) and, as such, the above field strengths are associated with $SU(2)_I$. The total electric and total magnetic field strengths may be written from equations (77) and (78) as

$$\mathbf{E} = -\nabla\phi_e^3 - \nabla\phi_e^8 + e\phi_e^3(A_1)^2 - (\nabla\times\mathbf{B}_1) \tag{79a}$$

and

$$\mathbf{H} = -\nabla\phi_g^3 - \nabla\phi_g^8 + g\phi_g^3(B_1)^2 + (\nabla\times\mathbf{A}_1) \tag{79b}$$

Case II. In this case all the A_k except A_4, A_5 and all B_k except B_4, B_5 vanish and the field equations are (58), (59). The nonvanishing components A_4, A_5 or B_4, B_5 may be identified as forming two members of U spin. The field strengths in this case would be associated with $SU(2)_U$.

Similar to Case I, we may set here $A_5 = 0 = B_5$; the field equations (58), (59) will then contain only A_4 and B_4, for which we may again assume the forms (64a) and (64b), respectively. The total charge is

$$Q = Q_e + Q_g \tag{80}$$

where

$$Q_e = \int \left[-\nabla^2\left(\frac{\phi_e^3+\sqrt{3}\,\phi_e^8}{2}\right) - \nabla^2\left(\frac{\phi_e^8-\sqrt{3}\,\phi_e^3}{2}\right) + e^2(A_4)^2 \frac{\phi_e^3+\sqrt{3}\,\phi_e^8}{2}\right] d^3x$$

$$= \int e^2(A_4)^2 \frac{\phi_e^3+\sqrt{3}\,\phi_e^8}{2}\, d^3x$$

$$= eC^2 I_1 \tag{81a}$$

where I_1 is given by (69). In this case equation (67a) reads

$$\frac{e(\phi_e^3+\sqrt{3}\,\phi_e^8)}{2} = \frac{1}{a} F(x, \theta) \tag{65a}$$

Similarly,

$$Q_g = \int g^2(B_4)^2 \frac{\phi_g^3+\sqrt{3}\,\phi_g^8}{2}\, d^3x$$

$$= gC'^2 I_1' \tag{81b}$$

where I_1' is described in equation (70); and (65b) is to be replaced by

$$\frac{g(\phi_g^3+\sqrt{3}\,\phi_g^8)}{2} = \frac{1}{a'} F(x', \theta') \tag{65b$'$}$$

The field equations (58)-(59) (setting $A_5 = 0 = B_5$) then tell us that in this case the sources $\frac{1}{2}(q_e^8 - \sqrt{3}\, q^3)$ and $\frac{1}{2}(q_g^8 - \sqrt{3}\, q_g^3)$ produce the Coulomb fields, while the electric source $\frac{1}{2}(q_e^3 + \sqrt{3}\, q_e^8)$ and the magnetic source $\frac{1}{2}(q_g^3 + \sqrt{3}\, 3q_g^8)$ get screened and produce magnetic and electric dipole fields, respectively. The electric field strengths are

$$E^{3e} = -\boldsymbol{\nabla}\left(\frac{\phi_e^3-\sqrt{3}\,\phi_e^3}{2}\right)$$

$$E^{8e} = -\boldsymbol{\nabla}\left(\frac{\phi_e^8+\sqrt{3}\,\phi_e^8}{2}\right)$$

$$E^5 = \frac{e(\phi_e^3+\sqrt{3}\,\phi_e^8)\mathbf{A}_4}{2}$$

$$= CF\left(\frac{r}{a}, \theta\right) f\left(\frac{r}{a}, \theta\right) \frac{(\hat{3}+\hat{8})\times\hat{4}}{r^3}$$

$$E^{4g} = -(\boldsymbol{\nabla}\times\mathbf{B}_4)$$

$$= -\frac{3(\mathbf{p}_3\cdot\mathbf{x}')\mathbf{x}' - \mathbf{p}_3 r'^3}{r'^5} f\left(\frac{r'}{a'}, \theta\right) - \frac{\mathbf{p}_3(\mathbf{x}'\cdot\boldsymbol{\nabla}f') - \mathbf{x}'(\mathbf{p}\cdot\boldsymbol{\nabla}f')}{r'^3} \tag{82}$$

and the magnetic field strengths are

$$H^{3g} = -\nabla\left(\frac{\phi_g^3 - \sqrt{3}\,\phi_g^3}{2}\right)$$

$$H^{8g} = -\nabla\left(\frac{\phi_g^8 + \sqrt{3}\,\phi_g^8}{2}\right)$$

$$H^{5g} = \frac{g(\phi_g^3 + \sqrt{3}\,\phi_g^8)\mathbf{B}_4}{2}$$

$$= C'F\left(\frac{r'}{a'}, \theta'\right) f\left(\frac{r'}{a'}, \theta'\right) \frac{(\hat{3}+\hat{8}) \times \hat{4}}{r'^3}$$

$$H^{4e} = (\nabla \times \mathbf{A}_4)$$

$$= \frac{3(\mathbf{m}_3 \cdot \mathbf{x}) - \mathbf{m}_3 r^3}{r^5} f\left(\frac{r}{a}, \theta\right) + \frac{\mathbf{m}_3(\mathbf{x} \cdot \nabla f) - \mathbf{x}(\mathbf{m}_3 \cdot \nabla f)}{r^3} \tag{83}$$

where $p_3 = Ca(\hat{3}+\hat{8})$ is the electric dipole moment and $m_3 = C'a'(\hat{3}+\hat{8})$ is the magnetic dipole moment. The fields E^{4g} and H^{4e} are long ranged. Combining them, the total electric and magnetic field strengths may be written as

$$\mathbf{E} = -\nabla\left(\frac{\phi_e^3 + \sqrt{3}\,\phi_e^8}{2}\right) - \nabla\left(\frac{\phi_e^8 - \sqrt{3}\,\phi_e^3}{2}\right)$$

$$+ \frac{e(\phi_e^3 + \sqrt{3}\,\phi_e^8)A_4}{2} - (\nabla \times \mathbf{B}_4) \tag{84a}$$

$$\mathbf{H} = -\nabla\left(\frac{\phi_g^3 + \sqrt{3}\,\phi_g^8}{2}\right) - \nabla\left(\frac{\phi_g^8 - \sqrt{3}\,\phi_g^8}{2}\right)$$

$$+ \frac{g(\phi_g^3 + \sqrt{3}\,\phi_g^8)}{2} + (\nabla \times \mathbf{A}_4) \tag{84b}$$

Case III. Similar to Cases I and II, we may easily obtain the results for this case as well. Since the field equations (60)–(61) retain only A_6, A_7 and B_6, B_7 as nonvanishing, they may form the two members of V spin. The field strengths in this case would then be associated with $SU(2)$.

The sources $\frac{1}{2}(\sqrt{3}\,q_e^3 + q_e^8)$ and $\frac{1}{2}(\sqrt{3}\,q_g^3 + q_g^8)$ produce the Coulomb fields, while the sources $\frac{1}{2}(q_e^3 - \sqrt{3}\,q_e^8)$ and $(q_g^3 - \sqrt{3}\,q_g^8)$ get screened and produce dipole fields. The total charge is

$$Q = Q_e + Q_g \tag{85a}$$

where

$$Q_e = eC^2 I_1 \tag{85b}$$

and

$$Q_g = gc'^2 I_1' \tag{85c}$$

where I_1 and I_1' are defined in equations (69) and (70), respectively. Equations (65a) and 65b) in this case read

$$e(\phi_e^3 - \sqrt{3}\,\phi_e^8) = \frac{1}{a} F(x, \theta) \tag{65a''}$$

$$g(\phi_g^3 - \sqrt{3}\,\phi_g^8) = \frac{1}{a'} F(x', \theta') \tag{65b''}$$

The electric field strengths are

$$E^{3e} = -\nabla\left(\frac{\phi_e^3 + \sqrt{3}\,\phi_e^3}{2}\right)$$

$$E^{8e} = -\nabla\left(\frac{\phi_e^8 - \sqrt{3}\,\phi_e^8}{2}\right)$$

$$E^{7e} = \frac{e(\phi_e^3 - \sqrt{3}\,\phi_e^8)}{2}$$

$$= CF\left(\frac{r}{a}, \theta\right) f\left(\frac{r}{a}, \theta\right) \frac{(\hat{3} - \hat{8}) \times \hat{6}}{r^3}$$

$$E^{6g} = -(\nabla \times \mathbf{B}_6)$$

$$= \frac{-3(\mathbf{p}_4 \cdot \mathbf{x}')\mathbf{x}' - \mathbf{p}_4 r'^3}{r'^5} f\left(\frac{r'}{a'}, \theta\right) - \frac{\mathbf{p}_4(\mathbf{x}' - \nabla f') - \mathbf{x}'(\mathbf{p}_4 - \nabla f')}{r'^3} \tag{86}$$

and the magnetic field strengths are

$$H^{3g} = -\nabla\left(\frac{\phi_g^3 + \sqrt{3}\,\phi_g^3}{2}\right)$$

$$H^{8g} = -\nabla\left(\frac{\phi_g^8 - \sqrt{3}\,\phi_g^8}{2}\right)$$

$$H^{7g} = g(\phi_g^3 - \sqrt{3}\,\phi_g^8)$$

$$= C'F\left(\frac{r'}{a'}, \theta\right) f\left(\frac{r'}{a'}, \theta\right) \frac{(\hat{3} - \hat{8}) \times \hat{6}}{r^3}$$

$$H^{6e} = (\nabla \times \mathbf{A}_6)$$

$$= \frac{3(\mathbf{m}_4 \cdot \mathbf{x}) - \mathbf{m}_4 r^3}{r^5} f\left(\frac{r}{a}, \theta\right) + \frac{\mathbf{m}_4(\mathbf{x} \cdot \nabla f) - \mathbf{x}(\mathbf{m}_4 \cdot \nabla f)}{r^3} \tag{87}$$

where $p_4 = Ca(\hat{3}-\hat{8})$ is the electric dipole moment and $m_4 = C'a'(\hat{3}-\hat{8})$ is the magnetic dipole moment. The fields E^{6g} and H^{6e} show the long-range behavior of the electric and magnetic dipole fields. The total electric and magnetic field strengths in this case are

$$\mathbf{E} = -\nabla\left(\frac{\phi_e^3 + \sqrt{3}\,\phi_e^8}{2}\right) - \nabla\left(\frac{\phi_e^8 - \sqrt{3}\,\phi_e^8}{2}\right) + \frac{e(\phi_e^3 - \sqrt{3}\,\phi_e^8)}{2} A_6 - (\nabla \times \mathbf{B}_6) \tag{88a}$$

$$\mathbf{H} = -\nabla\left(\frac{\phi_g^3 + \sqrt{3}\,\phi_g^8}{2}\right) - \nabla\left(\frac{\phi_g^8 - \sqrt{3}\,\phi_g^8}{2}\right) + \frac{g(\phi_g^3 - \sqrt{3}\,\phi_g^8)}{2} B_6 + (\nabla \times \mathbf{B}_6) \tag{88b}$$

6. ENERGY OF THE SOLUTIONS

Let us now calculate the energies of both the Coulomb and dipole solutions. We calculate them for the gauge group $SU(2)$ and from that infer the results for $SU(3)$ as well. For the energy of the Coulomb solutions we write

$$H^c = H_e^c + H_g^d \tag{89}$$

where H_e^c is the Coulomb energy contribution due to the electric sources and H_g^c is that due to the magnetic sources; they may be written as

$$H_e^c = \int \frac{Q_e^2}{r^3}\, d^3x \tag{90a}$$

$$H_g^c = \int \frac{Q_g^2}{r'^3}\, d^3x' \tag{90b}$$

where Q_e and Q_g represent the extended electric and magnetic charge distributions. Using $d^3x = 2\pi \int_0^\infty \int_0^\pi r^2\, dr \sin\theta\, d\theta$, $r = ax$, and $r' = a'x'$, we can write equations (90) in the form

$$H_g^c = \frac{Q_e^2}{a} I_2 \tag{91a}$$

$$H_g^c = \frac{Q_g^2}{a'} I_2' \tag{91b}$$

where I_2 and I_2' are integrals depending only on the shape of the charge distributions.

The energy corresponding to the dipole solutions may be calculated from equations (68)-(73) as

$$H^{\mathrm{d}}=H_e^{\mathrm{d}}+H_g^{\mathrm{d}} \tag{92}$$

where

$$H_e^{\mathrm{d}}=\int d^3x[\tfrac{1}{2}(\nabla\phi_e^3)^2+e^2(\phi_e^3A_1)^2+(\nabla\times A_1)^2] \tag{93a}$$

$$H_g^{\mathrm{d}}=\int d^3x[\tfrac{1}{2}(\nabla\phi_g^1)^2+g^2(\phi_g^1B_3)^2+(\nabla\times B_3)^2] \tag{93b}$$

which on integrating the last terms and using equations (46b) and (47b), respectively, may be written as

$$H_e^{\mathrm{d}}=\int d^3x[\tfrac{1}{2}(\nabla\phi_e^3)^2+e^2(\phi_e^3A_1)^2] \tag{94a}$$

$$H_g^{\mathrm{d}}=\int d^3x[\tfrac{1}{2}(\nabla\phi_g^1)^2+g^2(\phi_g^1B_3)^2] \tag{94b}$$

Now, looking at the energy (94a) and (94b) along with equations (64)-(67), we can show the dependence of the energy of the dipole solutions on the shape, the coupling parameters, the total charge, and the extensions a and a' of the respective sources. Using these equations, we can write equations (94a) and (94b) as

$$H_e^{\mathrm{d}}=\frac{1}{a}\left(\frac{I_3}{e^2}+\frac{Q_e}{e}\frac{I_4}{I_1}\right) \tag{95a}$$

$$H_g^{\mathrm{d}}=\frac{1}{a'}\left(\frac{I_3'}{g^2}+\frac{Q_g}{g}\frac{I_4'}{I_1'}\right) \tag{95b}$$

where I_1 is given by equations (69) and

$$I_3=2\pi\int_0^\pi \sin\theta\,d\theta\int_0^\infty x^2\,dx\left[\left(\frac{\partial F(x,\theta)}{\partial x}\right)^2+\frac{1}{x^2}\left(\frac{\partial F(x,\theta)}{\partial\theta}\right)^2\right] \tag{96a}$$

$$I_4=2\pi\int_0^\pi \sin\theta\,d\theta\int_0^\infty \frac{dx}{x^2}F^2(x,\theta)f^2(x,\theta) \tag{96b}$$

while I_1', I_3', and I_4' are the primed integrals of I_1, I_3, and I_4.

A comparison of equations (95a) and (91a) shows that while the energy of the magnetic dipole solutions (95a) is linear in Q_e, the Coulomb energy is quadratic in Q_e. Similar comparison may be made for the electric dipole solutions (95b) and equation (91b). Now, combining equations (91a) and (95a), we obtain the following equation quadratic in $(Q_e e)$:

$$(Q_e e)^2 I_1 I_2 H_e^{\rm d} - (Q_e e) I_4 H_e^{\rm c} - I_1 I_3 H_e^{\rm c} = 0 \tag{97a}$$

Similarly, combining equations (91b) and (95b) yields

$$(Q_g g)^2 I_1' I_2' H_g^{\rm d} - (Q_g g) I_4' H_g^{\rm c} - I_1' I_3' H_g^{\rm c} = 0 \tag{97b}$$

Solving equations (97), we obtain the values for $(Q_e e)$ and $(Q_g g)$,

$$(Q_e e) = \{I_4 H_e^{\rm c} \pm [I_4^2 (H_e^{\rm c})^2 + 4 I_1^2 I_2 I_3 H_e^{\rm c} H_e^{\rm d}]^{1/2}\}/2 I_1 I_2 H_e^{\rm d} \tag{98a}$$

$$(Q_g g) = \{I_4' H_g^{\rm c} \pm [I_4'^2 (H_g^{\rm c})^2 + 4 I_1'^2 I_2' I_3' H_g^{\rm c} H_g^{\rm d}]^{1/2}\}/2 I_1' I_2' H_g^{\rm d} \tag{98b}$$

Of two values each for $(Q_e e)$ and $(Q_g g)$, only the positive values are allowed, so that the energies (95) remain positive definite. It may also be observed that the values of $(Q_e e)$ and $(Q_g g)$ change according to the energies of the Coulomb and dipole solutions. Critical values of $(Q_e e)$ and $(Q_g g)$ may therefore be obtained from equations (98a) and (98b), respectively, for $H_e^{\rm c} = H_e^{\rm d}$ and for $H_g^{\rm c} = H_g^{\rm d}$, which give

$$(Q_e e)_{\rm critical} = \frac{1}{2I_2}\,\frac{I_2}{I_1} + \left[\left(\frac{I_2}{I_1}\right)^2 + 4 I_3 I_2\right]^{1/2} \tag{99a}$$

$$(Q_g g)_{\rm critical} = \frac{1}{2I_2'}\,\frac{I_2'}{I_1'} + \left[\left(\frac{I_2'}{I_1'}\right)^2 + 4 I_3' I_2'\right]^{1/2} \tag{99b}$$

further,

$$\text{if} \quad H_e^{\rm d} < H_e^{\rm c}, \qquad (Q_e e) > (Q_e e)_{\rm critical} \tag{99c}$$

$$\text{if} \quad H_e^{\rm d} > H_e^{\rm c}, \qquad (Q_e e) < (Q_e e)_{\rm critical} \tag{99d}$$

and similar results for $(Q_g g)$ with respect to $H_g^{\rm c}$ and $H_g^{\rm d}$. Therefore, it may be concluded that if

$$[(Q_e e) + (Q_g g)] > [(Q_e e) + (Q_g g)]_{\rm critical} \tag{100a}$$

then

$$H^{\rm d} < H^{\rm c} \tag{100b}$$

where $H^{\rm d}$ and $H^{\rm c}$ are given by equations (92) and (89), respectively. These equations therefore fix a threshold value for the product of charge and coupling parameters, above which the energy corresponding to the dipole solutions would be lower than the energy corresponding to the Coulomb solutions.

For the gauge group $SU(3)$, similar calculations may be carried out for the three cases and it may be concluded here also that for sufficiently large electric and magnetic source strengths the energy of the dipole solutions becomes lower than the energy of the corresponding Coulomb solutions.

7. DISCUSSION

Incorporating magnetic sources through a new non-Abelian field tensor [equation (1)], we have studied the Sikivie–Weiss magnetic dipole solutions for the gauge groups $SU(2)$ and $SU(3)$. The conclusions of course extend to higher gauge groups as well. Due to the presence of both electric and magnetic sources, we have called them dipole solutions. It has been observed that the field tensor (1) allows the incorporation of magnetic sources on the lines of the electric sources, expect that due to the negative sign in the corresponding field equation, the cylindrical symmetry is about the opposite axes than for the electric sources. The obtained solutions are in fact of three kinds: Coulomb, totally screened, and partially screened ones. The partially screened solutions alone have been shown to have the long-range behavior of dipole fields and their energy has been compared with that of the Coulomb fields. The comparison leads us to the conclusion that the results of Sikivie and Weiss remain valid even if we introduce magnetic sources in the theory.

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